

# Dynamics of isolated left orders

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## 1. Introduction

Throughout this paper, all the groups considered are countable. Given a group  $G$ , a total order  $<_\lambda$  on  $G$  is called a *left order* if for any  $f, g, h \in G$ ,  $f <_\lambda g$  implies  $hf <_\lambda hg$ . An element  $g \in G$  is called  $\lambda$ -positive if  $g >_\lambda e$ . The set of all the  $\lambda$ -positive elements is called the *positive cone of  $\lambda$*  and is denoted by  $P_\lambda$ . It is a semigroup and  $P_\lambda \sqcup P_\lambda^{-1} = G \setminus \{e\}$ .

Given a left order  $<_\lambda$ , we define  $\lambda : G \setminus \{e\} \rightarrow \{\pm 1\}$  by  $\lambda(g) = 1$  if and only if  $g \in P_\lambda$ . Then we have

$$(1.1) \quad \lambda(f) = 1, \lambda(g) = 1 \Rightarrow \lambda(fg) = 1, \text{ and } \lambda(f^{-1}) = -\lambda(f).$$

Conversely given a map  $\lambda : G \setminus \{e\} \rightarrow \{\pm 1\}$  which satisfies (1.1), we get a left order  $<_\lambda$  by setting  $f <_\lambda g$  if  $\lambda(f^{-1}g) = 1$ . The map  $\lambda$  is also referred to as a left order. Thus the set  $LO(G)$  of the left orders on  $G$  is viewed as a closed subset of the space  $\{\pm 1\}^{G \setminus \{e\}}$  with the pointwise convergence topology. This yields a totally disconnected compact metrizable topology on  $LO(G)$  (metrizable since  $G$  is countable). It is either finite or uncountably many [6]. We call  $\lambda \in LO(G)$  *isolated* if it is an isolated point in the space  $LO(G)$ .

Given  $\lambda \in LO(G)$ , there is defined a dynamical realization

$$\rho_\lambda : G \rightarrow \text{Homeo}^+(\mathbb{R})$$

based at  $x_0 \in \mathbb{R}$  such that  $f <_\lambda g$  if and only if  $fx_0 < gx_0$ . We discuss its fundamental properties in Section 2. Especially we show that the dynamical realization is tight at the base point. See Definition 2.1.

In this paper, we are mainly interested in isolated orders, since in this case, the dynamical realizations display a certain kind of rigidity, and vice versa. See [8] for the case of circular orders.

An action  $\rho : G \rightarrow \text{Homeo}_+(\mathbb{R})$  is said to be *cocompact* if there is a compact interval  $I$  such that any orbit  $\rho(G)x$  intersects  $I$ . Our first result, proved in Section 3, is the following.

**THEOREM 1.** *If  $\lambda \in LO(G)$  is isolated, then its dynamical realization  $\rho_\lambda$  is cocompact.*

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By Theorem 1, the dynamical realization of an isolated order admits a minimal set  $\mathcal{M}$ , which is shown to be unique unless  $G \cong \mathbb{Z}$ . In Section 4, we show that if  $\mathcal{M} = \mathbb{R}$ , then the group is rational (Theorem 4.1).

Given  $\lambda \in LO(G)$ , a subgroup  $H$  of  $G$  is called  $\lambda$ -convex, if whenever  $h_1, h_2 \in H$ ,  $g \in G$  and  $h_1 <_\lambda g <_\lambda h_2$ , we have  $g \in H$ . The set of convex subgroups is totally ordered by the inclusion. The following theorem is shown in Section 5.

**THEOREM 2.** *If  $\lambda \in GO(G)$  is isolated, then there are finitely many convex subgroups.*

This enables us to define the maximal sequence of convex subgroups of an isolated left order. As an application of our method, we give a dynamical proof of the Tararin theorem which characterizes the groups with finitely many left orders in Section 6. In Section 7, the maximal Tararin subgroup of an isolated left order is defined, and is shown to be equal to the Conradian soul [10]. Many isolated left orders are induced from isolated circular orders of the group quotiented by the center. However there is an example of a group with isolated left orders which admits no center. This is given in Section 8.

Last sections 9 and 10 are more or less independent of the previous sections. Let  $G$  be a group with center  $Z(G)$  isomorphic to  $\mathbb{Z}$ . If  $Z(G)$  is  $\lambda$ -cofinal for some  $\lambda \in LO(G)$ ,  $\lambda$  induces a circular order  $c(\lambda)$  of the group  $G/Z(G)$ . See Section 9 for details.

**THEOREM 3.** *Under certain conditions, if  $c(\lambda)$  is isolated, then  $\lambda$  is isolated.*

See Theorem 9.13. Let  $B_3$  be the braid group of 3 strings. Dubrovina-Dubrovin [2] constructed an isolated order  $\lambda_3$  on  $B_3$ . In section 10, we show:

**THEOREM 4.** *There is an isolated order  $\lambda_M$  in  $LO(B^3)$  which is not an automorphic image of  $\lambda_3$ .*

For this, we find two isolated circular orders of  $B_3/Z(B_3) \cong PSL(2, \mathbb{Z})$  using a theorem of Matsuda [9] concerning the rotation number rigidity of the group  $PSL(2, \mathbb{Z})$ , and use Theorem 3.

## 2. Dynamical Realization

In this section, we define a dynamical realization of a left order  $\lambda \in LO(G)$  and study its fundamental properties. Fix an enumeration of  $G$ :  $G = \{g_i \mid i \in \mathbb{N}\}$  such that  $g_1 = e$ . We define an order preserving embedding  $\iota : G \rightarrow \mathbb{R}$  inductively as follows. Define  $\iota(g_1) = x_0$ , where  $x_0$  is some point in  $\mathbb{R}$ . Assume we have defined  $\iota$  on the subset  $\{g_1, \dots, g_n\}$ ,  $n \geq 1$ , and let us define  $\iota(g_{n+1})$ . Order the subset  $\{g_1, \dots, g_n\}$  as

$$g_{i_1} <_\lambda g_{i_2} <_\lambda \dots <_\lambda g_{i_n}.$$

If  $g_{n+1} <_\lambda g_{i_1}$ , define  $\iota(g_{n+1}) = \iota(g_{i_1}) - 1$ ,

if  $g_{i_n} <_\lambda g_{n+1}$ ,  $\iota(g_{n+1}) = \iota(g_{i_n}) + 1$ ,

and if  $g_{i_k} <_\lambda g_{n+1} <_\lambda g_{i_{k+1}}$ ,  $\iota(g_{n+1}) = (1/2)(\iota(g_{i_k}) + \iota(g_{i_{k+1}}))$ .

Then we have  $\inf \iota(G) = -\infty$  and  $\sup \iota(G) = \infty$ . The left translation of  $G$  yields an order preserving action of  $G$  on  $\iota(G)$ , which extends to a continuous action on the closure  $\text{Cl}(\iota(G))$ . Extend it further to a continuous action on  $\mathbb{R}$  by setting that the action on gaps of  $\text{Cl}(\iota(G))$  be linear.

This action is called the *dynamical realization of  $\lambda$  based at  $x_0$* , and is denoted by  $\rho_\lambda$ . The dynamical realization depends on the choice of the enumeration of  $G$ . Soon later, we shall show that any one of them are mutually topologically conjugate.

DEFINITION 2.1. An action  $\rho : G \rightarrow \text{Homeo}_+(\mathbb{R})$  is called *tight at  $x_0 \in \mathbb{R}$*  if

- (1)  $\rho$  is free at  $x_0$  i.e, the stabilizer at  $x_0$  is trivial,
- (2)  $\inf \rho(G)x_0 = -\infty$ ,  $\sup \rho(G)x_0 = \infty$ , and
- (3) If  $\text{Cl}(\rho(G)x_0) \cap [a, b] = \{a, b\}$  for any  $a < b$ , then  $\{a, b\} \subset \rho(G)x_0$ .

PROPOSITION 2.2. *The dynamical realization  $\rho_\lambda$  based at  $x_0$  is tight at  $x_0$ .*

PROOF. All that needs proof is (3). Let  $a < b$  be as in (3). The proof is by contradiction. Assume, to fix the idea, that  $a \notin \rho(G_0)x_0 = \iota(G)$ . (Notice that  $\iota(g) = \rho_\lambda(g)x_0$ .) Choose  $\epsilon$  small enough compared with  $b - a$ , and choose  $\iota(g_1) \in (a - \epsilon, a)$  and  $\iota(g_2) \in [b, b + \epsilon)$ . Recall that the dynamical realization is defined via an enumeration of  $G$ . One may assume that there is no point in  $(\iota(g_1), \iota(g_2)) \cap \iota(G)$  which is enumerated before  $g_1$  or  $g_2$ , since otherwise one may pass to that point. Since  $a \in \text{Cl}(\iota(G)) \setminus \iota(G)$ , there is a point  $\iota(g_3)$  in  $(\iota(g_1), \iota(g_2)) \cap \iota(G)$  which is enumerated for the first time after  $g_1$  and  $g_2$ . Then  $\iota(g_3)$  is the midpoint of  $\iota(g_1)$  and  $\iota(g_2)$  and must be fallen in  $(a, b)$  since  $\epsilon$  is small. A contradiction.  $\square$

COROLLARY 2.3. *The dynamical realizations defined via two different enumerations of  $G$  are mutually conjugate by an orientation and base point preserving homeomorphism of  $\mathbb{R}$*

PROOF. Let  $\iota$  and  $\iota'$  be two embeddings of  $G$  obtained by different enumerations of  $G$ . There is an orientation preserving bijection  $h : \iota(G) \rightarrow \iota'(G)$  defined by  $h(\iota(g)) = \iota'(g)$  ( $g \in G$ ). By the tightness,  $h$  extends, first of all, to a homeomorphism  $h : \text{Cl}(\iota(G)) \rightarrow \text{Cl}(\iota'(G))$ , and then to a homeomorphism of  $\mathbb{R}$  linearly on gaps. The extended  $h$  yield the required conjugacy.  $\square$

A left order  $<_\lambda$  is called *discrete* if there is a minimal  $\lambda$ -positive element, and *indiscrete* otherwise.

COROLLARY 2.4. *If  $\lambda \in LO(G)$  is indiscrete, then the orbit  $\rho_\lambda(G)x_0$  of the base point  $x_0$  is dense in  $\mathbb{R}$ .*

PROOF. Assume  $\text{Cl}(\rho_\lambda(G)x_0) \neq \mathbb{R}$  and let  $(a, b)$  be a gap of  $\text{Cl}(\rho_\lambda(G)x_0)$ . Then by the previous lemma, we have  $a, b \in \rho_\lambda(G)x_0$ . That is,  $a = \iota(g_1)$  and  $b = \iota(g_2)$ . Then  $g_1^{-1}g_2$  is the minimal positive element, and  $\lambda$  is discrete.  $\square$

### 3. Proof of Theorem 1

We begin with two lemmas.

LEMMA 3.1. *Let  $G$  be a finitely generated group which acts on  $\mathbb{R}$  without global fixed points. Then the action is cocompact.*

PROOF. We identify  $\mathbb{R} \approx (0, 1)$ . Let  $G_0$  be a finite generating set of  $G$ . Define

$$a = \sup_{s \in G_0} \sup_{x \in (0, 1)} |sx - x|.$$

Choose a compact interval  $J \subset (0, 1)$  such that  $|J| > a$ . Given any point  $x \in (0, 1)$ , we have  $\inf Gx = 0$  and  $\sup Gx = 1$  since there is no global fixed point. Considering the Schreier graph of  $Gx$ , one can show that  $Gx \cap J \neq \emptyset$ .  $\square$

LEMMA 3.2. *Let  $G$  be a group acting on  $\mathbb{R}$  and let  $y_0 \in \mathbb{R}$ . Denote by  $G_{y_0}$  the stabilizer of  $G$  at  $y_0$ . Given  $\lambda_0 \in LO(G_{y_0})$ , there are at least two orders in  $LO(G)$  which restrict to  $\lambda_0$  on  $G_{y_0}$ .*

PROOF. Let  $\mu$  be the  $G$ -invariant order on  $G/G_{y_0}$  given by the order of the orbit  $Gy_0 \approx G/G_{y_0}$ . Then  $\lambda_0$  and  $\mu$  determines a left order on  $G$  lexicographically (Lemma 5.1). If we consider the reciprocal order  $-\mu$ , we get another one.  $\square$

Assume  $\lambda \in LO(G)$  is an isolated left order on  $G$ . Since we are considering the poitwise convergence topology, this is equivalent to the following condition  $(\star)$

$(\star)$  *There is a finite subset  $S \subset P_\lambda$  such that  $\lambda$  is the only element in  $LO(G)$  which contains  $S$  in its positive cone.*

Such a subset  $S$  is called a *characteristic positive set* of  $\lambda$ .

PROOF OF THEOREM 1. By the dynamical realization of the isolated left order  $\lambda$ , the group  $G$  acts on  $\mathbb{R}$ . Let  $H$  be the subgroup of  $G$  generated by a characteristic positive set  $S$  of  $\lambda$ . If there is no global fixed point by the action of  $H$ , then  $H$  acts on  $\mathbb{R}$  cocompactly, and hence also  $G$ , finishing the proof. In the remaining case, choose a global fixed point  $y_0$  of  $H$  and consider  $G_{y_0}$ . By the previous lemma, the restriction of  $\lambda$  to  $G_{y_0}$  extends to two left orders of  $G$ . But we have  $S \subset H \subset G_{y_0}$  and hence  $S$  is contained in the positive cone of both orders. A contradiction.  $\square$

REMARK 3.3. The condition that  $\lambda$  be isolated is actually necessary for Theorem 1. To show this, let  $G$  be the infinite direct sum of  $\mathbb{Z}$ , i.e,

$$G = \{(a_n)_{n \in \mathbb{N}} \mid a_n \in \mathbb{Z}, \ a_n = 0 \text{ but for finitely many } n\}.$$

Define a left order on  $G$  by setting  $0 < (a_n)$  if  $0 < a_N$ , where  $N$  is the largest number such that  $a_N \neq 0$ . Then its dynamical realization is not cocompact. To show this, define for  $m \in \mathbb{N}$ ,

$$G_m = \{(a_n) \mid a_n = 0, \ \forall n > m\}.$$

Then  $G_m$ 's form an exhausting increasing sequence of convex subgroups. Consider the dynamical realization  $\rho_\lambda$  based at  $x_0$ . The points

$$\xi_n = \inf \rho_\lambda(G_n)x_0 \quad \text{and} \quad \eta_n = \sup \rho_\lambda(G_n)x_0$$

are fixed points of  $\rho_\lambda(G_n)$ . They satisfy  $\xi_n \searrow -\infty$  and  $\eta_n \nearrow \infty$  by condition (2) of Definition 2.1, since  $G_n$  is exhausting. This implies that  $\rho_\lambda$  is not cocompact.

Theorem 1 implies that there is a minimal set  $\mathcal{M}$  for the dynamical realization of an isolated left order. There are trichotomy for  $\mathcal{M}$  ([3] Proposition 6.1).

(I)  $\mathcal{M} = \mathbb{R}$ .

(II)  $\mathcal{M}$  is discrete in  $\mathbb{R}$ .

(III)  $\mathcal{M}$  is locally Cantor. In this case, if  $X$  is a nonempty closed subset of  $\mathbb{R}$  invariant by the dynamical realization of  $G$ , then  $\mathcal{M} \subset X$ . Especially,  $\mathcal{M}$  is the unique minimal set.

LEMMA 3.4. *Let  $\lambda \in LO(G)$  be isolated, with  $\mathcal{M}$  the associated minimal set. Assume (III) above, or (II) and  $G \not\cong \mathbb{Z}$ . Then the base point  $x_0$  is contained in a gap  $I_1$  of  $\mathcal{M}$ , the stabilizer  $G_{I_1}$  is nontrivial, and there is no gap of  $\mathcal{M}$  other than the orbit of  $I_1$ .*

PROOF. We give a proof only for case (III), the other case being easier. Assume that the base point  $x_0$  is contained in  $\mathcal{M}$  and let  $(a, b)$  be a gap of  $\mathcal{M}$ . Since the dynamical realization  $\rho_\lambda$  is tight, we have  $a, b \in \rho_\lambda(G)x_0$ . But there is no orientation preserving homeomorphism leaving  $\mathcal{M}$  invariant and mapping  $a$  to  $b$ . The contradiction shows that  $x_0$  is contained in a gap  $I_1$  of  $\mathcal{M}$ .

If  $G_{I_1}$  is trivial, then  $\rho_\lambda(G) \cap I_1 = \{x_0\}$ . Again by the tightness, the boundary points of  $I_1$  must belong to  $\rho_\lambda(G)x_0$ . A contradiction. The last statement follows similarly from the tightness.  $\square$

COROLLARY 3.5. *If  $G \not\cong \mathbb{Z}$ , then the minimal set  $\mathcal{M}$  of the dynamical realization is unique.*

PROOF. All that needs proof is the case where  $\mathcal{M}$  is discrete. We still use the notation of the previous lemma. The stabilizer  $G_{I_1}$  is the maximal  $\lambda$ -convex subgroup (See Lemma 5.4), and hence determined only by the left order  $\lambda$ . This shows that the gap of the minimal set is determined by  $\lambda$ , and hence the minimal set is unique.  $\square$

#### 4. The case $\mathcal{M} = \mathbb{R}$

This section is devoted to the proof of the following theorem.

THEOREM 4.1. *Let  $\lambda \in LO(G)$  be isolated and assume that the dynamical realization  $\rho_\lambda$  is minimal. Then the group  $G$  is isomorphic to an additive subgroup  $A$  of  $\mathbb{Q}$  such that  $A \not\cong \mathbb{Z}$ , and  $\rho_\lambda$  is conjugate to the translation by  $A$  or by  $-A$ .*

Let  $\lambda$  be an element of  $LO(G)$  which satisfies the hypothesis of Theorem 4.1. We shall abbreviate the notations  $\rho_\lambda(g)x$  by  $gx$ , and  $\rho_\lambda(G) \subset \text{Homeo}_+(\mathbb{R})$  by  $G$ . Let  $Z$  be a centralizer of  $G$  in  $\text{Homeo}_+(\mathbb{R})$ .

LEMMA 4.2. *The centralizer  $Z$  is an abelian group which acts freely on  $\mathbb{R}$ .*

PROOF. For  $\zeta \in Z \setminus \{\text{id}\}$ ,  $\text{Fix}(\zeta)$  is a closed set which is invariant by  $G$ . Since the  $G$ -action is minimal, we have  $\text{Fix}(\zeta) = \emptyset$ . By Hölder's theorem (e.g, [10]), any group acting freely on  $\mathbb{R}$  is abelian.  $\square$

Let  $x_0$  be the base point of the dynamical realization. Choose  $x_n \in Gx_0$ ,  $n \in \mathbb{N}$ , such that  $x_n \rightarrow x_0$ . Notice that  $G$  acts freely at  $x_n$ . Let  $\lambda_n \in LO(G)$  be the order determined by  $x_n$ :  $g >_{\lambda_n} e$  if and only if  $gx_n > x_n$ . Then  $\lambda_n \rightarrow \lambda$  in  $LO(G)$ . Since  $\lambda$  is isolated,  $\lambda_n = \lambda$  for any large  $n$ . We assume  $\lambda_n = \lambda$  for all  $n$ . Define an order preserving bijection  $\zeta_n : Gx_0 \rightarrow Gx_n$  by  $\zeta_n(gx_0) = gx_n$ . Since  $Gx_0 = Gx_n$  is dense in  $\mathbb{R}$ , the map  $\zeta_n$  extends to an orientation preserving homeomorphism of  $\mathbb{R}$ , denoted by the same letter  $\zeta_n$ .

LEMMA 4.3. *We have  $\zeta_n \in Z$ .*

PROOF. Given any  $g \in G$ , it suffices to show that  $\zeta_n g = g \zeta_n$  on the dense subset  $Gx_0$ . For any  $hx_0 \in Gx_0$ , we have

$$\zeta_n g(hx_0) = \zeta_n((gh)x_0) = ghx_n = g(hx_n) = g\zeta_n(hx_0),$$

as is required.  $\square$

LEMMA 4.4. *The action of  $Z$  is minimal, and is conjugate to translations.*

PROOF. By Lemmas 4.2 and 4.3, there is an element in  $Z$  which acts freely on  $\mathbb{R}$ . This implies that the action of  $Z$  is cocompact. Let  $\mathcal{N}$  be a minimal set of  $Z$ . If it is locally Cantor, then  $\mathcal{N}$  is the unique minimal set, and must be invariant by  $G$ . But  $G$ -action is minimal by the assumption. A contradiction. Next assume  $\mathcal{N}$  is discrete. Then since the  $Z$ -action is free, we must have  $Z \cong \mathbb{Z}$ , contradicting Lemma 4.3. Therefore  $Z$  must act minimally on  $\mathbb{R}$ .

Choose any  $\zeta_0 \in Z \setminus \{\text{id}\}$ . Since the action of the group  $\langle \zeta_0 \rangle$  is free and  $Z$  is abelian, the group  $Z/\langle \zeta_0 \rangle$  acts on  $\mathbb{R}/\langle \zeta_0 \rangle \approx S^1$ . Since  $Z/\langle \zeta_0 \rangle$  is amenable, there is an  $Z/\langle \zeta_0 \rangle$ -invariant probability measure. It lifts to a locally finite  $Z$ -invariant measure  $\mu$  on  $\mathbb{R}$ . Since the action of  $Z$  is minimal,  $\mu$  is atomless and fully supported. Thus there is a homeomorphism  $h$  such that  $h_*\mu$  is the Lebesgue. Conjugating the  $Z$ -action by  $h$ , we obtain an action by translations.  $\square$

PROOF OF THEOREM 4.1. By changing the coordinate, we assume that the action of  $Z$  is by translations. Since the  $Z$ -action is minimal, any element of  $G$ , commuting with  $Z$ , acts also by translations. Then we have an injective homomorphism  $\phi : G \rightarrow \mathbb{R}$  defined by the translation length. We shall show that  $\phi$  embeds  $G$  into  $\mathbb{Q}$ . Assume not. Then  $G$  is a nontrivial direct sum:  $G = G_1 \oplus G_2$ . Given any  $a \in \mathbb{R}$ , we obtain a homomorphism  $\phi_a : G \rightarrow \mathbb{R}$  by setting  $\phi_a = \phi$  on  $G_1$  and  $\phi_a = a\phi$  on  $G_2$ . There is  $a$  arbitrarily near 1 such that  $\phi_a$  is injective. But  $\phi_a$  yields a left order different from  $\lambda$  and arbitrarily near  $\lambda$ . This contradicts the assumption that  $\lambda$  is isolated. We have shown that  $G$  is isomorphic to an additive subgroup  $A$  of  $\mathbb{Q}$ , and the action is by translations. It is easy to show that the action is conjugate to the translation by  $A$  or by  $-A$ .  $\square$

## 5. Convex subgroups

We shall prove Theorem 2 in this section. First we begin with fundamental properties of convex subgroups. We begin with a well known easy fact.

LEMMA 5.1. *Let  $H$  be a subgroup of  $G$ . For any  $\lambda_0 \in LO(H)$  and a  $G$ -invariant total order  $\lambda_1$  on  $G/H$ , there is a unique order  $\lambda \in LO(G)$  such that  $H$  is  $\lambda$ -convex, that  $\lambda|_H = \lambda_0$ , and that for  $g \notin H$ ,  $g >_\lambda e$  if and only if  $gH >_{\lambda_1} H$ .*  $\square$

Such an order  $\lambda$  is said to be *determined lexicographically* by  $\lambda_0$  and  $\lambda_1$ .

LEMMA 5.2. *Let  $\lambda \in LO(G)$  and  $H$  a  $\lambda$ -convex subgroup of  $G$ . Then there is a  $G$ -invariant total order  $\lambda_1$  on  $G/H$  such that  $\lambda$  is determined lexicographically by  $\lambda|_H$  and  $\lambda_1$ .*

PROOF. Define a total order  $\lambda_1$  on  $G/H$  by setting  $g_1H <_{\lambda_1} g_2H$  if  $e <_\lambda g_1^{-1}g_2$  and  $g_1^{-1}g_2 \notin H$ . The convexity of  $H$  shows that this is a well defined  $G$ -invariant order.  $\square$

Henceforth in this section we work under the following assumption.

ASSUMPTION 5.3. (1)  $\lambda \in LO(G)$  is isolated with a characteristic positive set  $S$ .

(2)  $G$  is not isomorphic to  $\mathbb{Z}$ .

(3) The minimal set  $\mathcal{M}$  of the dynamical realization is not  $\mathbb{R}$ .

Denote by  $I_1 = (y_0, z_0)$  the gap of  $\mathcal{M}$  which contains the base point  $x_0$  (Lemma 3.4), and by  $G_1$  the stabilizer of  $I_1$ .

- LEMMA 5.4. (1)  $G_1$  is proper and nontrivial.  
 (2)  $G_1$  is the maximal  $\lambda$ -convex subgroup of  $G$ .  
 (3) The restricted order  $\lambda|_{G_1}$  is isolated with characteristic positive set  $S \cap G_1$ .  
 (4)  $S \cap (G \setminus G_1) \neq \emptyset$ .

PROOF. The subgroup  $G_1$  is clearly proper. It is nontrivial by Lemma 3.4. Also  $G_1$  is convex. Let  $H$  be an arbitrary proper  $\lambda$ -convex subgroup of  $G$ . We shall show that  $H \subset G_1$ . Consider first the case where  $\mathcal{M}$  is discrete. By looking at the action of  $G$  on  $\mathcal{M}$ , one can define a surjective homomorphism  $\phi : G \rightarrow \mathbb{Z}$  such that  $\text{Ker}(\phi) = G_1$ . If  $\phi(H)$  is nontrivial, then clearly we have  $H = G$  since  $H$  is convex. If  $\phi(H)$  is trivial, then  $H \subset G_1$ , as is required.

So in the rest, we assume that  $\mathcal{M}$  is locally Cantor. Let  $\mathcal{H}$  be the convex hull of  $Hx_0$  in  $\mathbb{R}$ , an open subset of  $\mathbb{R}$ . The convexity of  $H$  implies that for any  $g \in G$ , we have either  $g\mathcal{H} = \mathcal{H}$  or  $g\mathcal{H} \cap \mathcal{H} = \emptyset$ . Thus the closed set

$$X = \mathbb{R} \setminus \bigcup_{g \in G} g\mathcal{H}$$

is  $G$ -invariant and nonempty. Therefore we have  $\mathcal{M} \subset X$ , which implies  $\mathcal{H} \subset I_1$ , showing that  $H \subset G_1$ .

Let us show that  $S \cap G_1$  is a characteristic positive set of  $\lambda|_{G_1}$ . If not, there is a left order  $\lambda'_0$  ( $\lambda'_0 \neq \lambda|_{G_1}$ ) of  $G_1$  such that  $S \cap G_1$  is contained in the positive cone of  $\lambda'_0$ . Let  $\lambda_1$  be the  $G$ -invariant total order on  $G/G_1$  obtained by Lemma 5.2. Let  $\lambda' \in LO(G)$  be the order determined lexicographically by  $\lambda'_0$  and  $\lambda_1$ . Then  $\lambda'$  contains  $S$  in its positive cone and  $\lambda' \neq \lambda$ , contradicting that  $S$  is a characteristic positive set of  $\lambda$ .

Finally let us show that  $S \cap (G \setminus G_1)$  is nonempty. If it is empty, then  $\lambda|_{G_1}$  and  $-\lambda_1$  lexicographically determines  $\lambda' \in LO(G)$ , where  $-\lambda_1$  is the reciprocal of the order  $\lambda_1$  constructed in Lemma 5.2. But  $S$  is contained in the positive cone of  $\lambda'$ . A contradiction.  $\square$

PROOF OF THEOREM 2. By Lemma 5.4, we obtain the maximal convex subgroup  $G_1$ . If  $G_1$  is not isomorphic to  $\mathbb{Z}$  or the minimal set of  $\lambda|_{G_1}$  is not the whole  $\mathbb{R}$ , then we can repeat the process and obtain the second maximal convex subgroup  $G_2$ . This process ends at finite steps since each time we lose the number of characteristic positive set.  $\square$

DEFINITION 5.5. The sequence

$$G = G_0 > G_1 > \cdots > G_n > \{e\}$$

of all the  $\lambda$ -convex subgroups is called the *maximal convex sequence* of the isolated order  $\lambda$ . The number  $n$  is called the *height* of  $\lambda$ .

Thus a left order with minimal dynamical realization has height 0. Let  $\mathcal{M}_0$  be the minimal set of  $G$  and  $I_1$  the gap of  $\mathcal{M}_0$  containing the base point  $x_0$ . Then  $G_1$  is the stabilizer of  $I_1$ . Let  $\mathcal{M}_1$  be the minimal set of the  $G_1$ -action on  $I_1$ , and  $I_2$  the gap of  $\mathcal{M}_1$  in  $I_1$  containing  $x_0$ . Continuing this way, we get a decreasing sequence of open intervals

$$\mathbb{R} \supset I_1 \supset \cdots \supset I_n.$$

Each subgroup  $G_i$  is the stabilizer of  $I_i$ , and each  $\mathcal{M}_i$  is a minimal set of  $G_i$  in  $I_i$ . The pair  $(I_i, \mathcal{M}_i)$  is called the  *$i$ -th internal pair associated with the maximal convex sequence*. There are only two possibilities for the last group  $G_n$ :



- (A)  $\mathcal{M}_n = I_n$ ,
- (B)  $G_n = \mathbb{Z}$ .

In (A), the order  $\lambda$  is indiscrete and in (B), it is discrete.

## 6. Tararin groups

DEFINITION 6.1. A group  $H$  is called a *Tararin group* if  $|LO(H)| < \infty$ .

Of course any order of a Tararin group is isolated. In this section, we shall give a dynamical proof of the following theorem by Tararin [11], [12]. See also [7].

THEOREM 6.2. (I) Assume  $|LO(G)| < \infty$ . Then the following holds.

(1) There is a unique rational series<sup>1</sup>

$$(6.1) \quad G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{e\}.$$

(The uniqueness implies that each subgroup  $G_i$  is characteristic, i.e., invariant by any automorphism of  $G$ . Especially it is a normal subgroup of  $G$ .)

(2) There are elements  $s_i \in G_i \setminus G_{i+1}$  for each  $i \in \{0, 1, \dots, n\}$  such that for any map  $\epsilon : \{0, 1, \dots, n\} \rightarrow \{\pm 1\}$ , there are exactly one order  $\lambda_\epsilon$  such that  $s_i^{\epsilon(i)}$  is positive. Moreover

$$LO(G) = \{\lambda_\epsilon \mid \epsilon \in \{\pm 1\}^{\{0, 1, \dots, n\}}\}.$$

(3) The sequence (6.1) is the maximal convex sequence for any  $\lambda_\epsilon$ .

(4) The quotient group  $G_i/G_{i+2}$ ,  $i \in \{0, \dots, n-1\}$ , is not bi-orderable.

(II) Conversely, if a group  $G$  admits a rational series (6.1) of normal subgroups  $G_i$  of  $G$  such that (4) holds, then  $|LO(G)| < \infty$ .

PROOF. First we shall show (I). Assume  $|LO(G)| < \infty$ . We shall prove the following claim by the induction on the height  $n$  of an order  $\lambda \in LO(G)$ .

The maximal convex sequence of  $\lambda$

$$(6.2) \quad G = G_0 > G_1 > \cdots > G_n > G_{n+1} = \{e\}$$

is a rational series, and there is a characteristic positive set of  $\lambda$  consisting of one element from each  $G_i \setminus G_{i+1}$ ,  $i = 0, \dots, n$ .

For  $n = 0$ , the claim follows from Theorem 4.1. For  $n \geq 1$ , assume it is true for  $n - 1$ . Thus the subsequence of (6.2) beginning with  $G_1$  is rational and there is a characteristic set  $S = \{t_1, \dots, t_r, s_1, \dots, s_n\}$  of  $\lambda$ , where  $t_j \in G \setminus G_1$  and  $s_i \in G_i \setminus G_{i+1}$ . We shall show that  $G_1$  is a normal subgroup of  $G$ , and  $G/G_1$  is a rational group. If we show this, the elements  $t_j$  can be replaced by a single element  $s_0 \in G \setminus G_1$ , and the proof of the claim will be complete. But this is clear if the minimal set  $\mathcal{M}$  of the dynamical realization  $\rho_\lambda$  is discrete.

So assume  $\mathcal{M}$  is a Cantor set. Let  $x_0$  be the base point of  $\rho_\lambda$ , and choose  $g_k \in G$  so that  $\rho_\lambda(g_k)x_0 \rightarrow \exists y_0 \in \mathcal{M}$  as  $k \rightarrow \infty$ . One may assume that  $\rho_\lambda(g_k)x_0$  belongs to a distinct gap of  $\mathcal{M}$  for each  $k$ . The left orders of  $G$  induced by the  $\rho_\lambda(G)$ -orbit of  $\rho(g_k)x_0$  are finite in number. So one may assume, by passing to a subsequence if necessary, that the left orders are the same. Of course this order is isolated. By the same argument as in Theorem 4.1, one can construct order preserving homeomorphisms  $h_{k,k'}$  of  $\text{Cl}(\rho_\lambda(G))x_0$  which commute with any  $\rho_\lambda(g)$

<sup>1</sup>Rational series means that for any  $i$ ,  $G_i/G_{i+1}$  is a rational group, i.e., an abelian group embeddable into  $\mathbb{Q}$ .



such that  $h_{k,k'}(\rho_\lambda(g_k)x_0) = \rho_\lambda(g_{k'})x_0$ . The map  $h_{k,k'}$  leaves the unique minimal set  $\mathcal{M}$  of  $\rho_\lambda(G)$  invariant.

Consider the quotient space  $\mathcal{R}$  of  $\mathbb{R}$  obtained by collapsing each gap of  $\mathcal{M}$  to a point. Then  $h_{k,k'}$  induces an order preserving homeomorphism of  $\mathcal{R}$  commuting with the induced action of  $\rho_\lambda(G)$ . Let  $Z$  be the centralizer of the action on  $\mathcal{R}$  induced from  $\rho_\lambda(G)$  in the space of the order preserving homeomorphisms of  $\mathcal{R}$ . Then since the induced action of  $G$  is minimal,  $Z$  acts freely on  $\mathcal{R}$ . In fact, if an element of  $Z$  fixes a point, then it must fix all the points in the  $G$ -orbit. Thus the action of  $Z$  is topologically conjugate to translations.

By the choices of  $k, k'$ , there are arbitrarily small translations. That is, the action must be minimal. This shows that the induced  $G$ -action on  $\mathcal{R}$  is also by translations. Therefore  $G_1$  is the kernel of the induced  $G$ -action, and is a normal subgroup of  $G$ . Finally, the order of  $G/G_1$  induced by  $\lambda$  must be isolated, and hence by Theorem 4.1,  $G/G_1$  is rational. Since  $G/G_1$  is rational, the elements  $t_1, \dots, t_r \in S \cap (G \setminus G_1)$ , which determine the order of  $G/G_1$  can be replaced by a single element  $s_0$ . This completes the proof of the claim.

Next we shall show that the cardinality of  $LO(G)$  is  $2^{n+1}$ , and (6.2) is the unique rational series of  $G$ . Let  $\lambda$  be as above, and let  $S = \{s_0, \dots, s_n\}$  be a characteristic positive set such that  $s_i \in G_i \setminus G_{i+1}$ . For any  $\epsilon : S \rightarrow \{\pm 1\}$ , define

$$S^\epsilon = \{s_i^{\epsilon(s_i)} \mid i = 0, \dots, n\}.$$

For any  $\epsilon$ , we can construct an order whose positive cone contains  $S^\epsilon$  lexicographically by the sequence (6.2). On the other hand, given an arbitrary order  $\lambda'$ ,  $s_i$  is either  $\lambda'$ -positive or  $\lambda'$ -negative. That is, there is  $\epsilon$  such that  $S^\epsilon$  is contained in the positive cone of  $\lambda'$ . This shows that the cardinality of  $LO(G)$  is  $2^{n+1}$ . Also a rational series of  $G$  is unique. In fact, any such series gives birth to a left order lexicographically. The series is the maximal convex sequence of that order, and the order is one of  $2^{n+1}$  orders. But all the orders have (6.2) as the maximal convex sequence.

The uniqueness of the rational series shows that the subgroup  $G_i$  in (6.2) is a characteristic subgroup, in particular a normal subgroup of  $G$ . We shall show that  $H = G_i/G_{i+2}$  is not bi-orderable. Denote  $A = G_{i+1}/G_{i+2}$  and  $B = G_i/G_{i+1}$ . Thus  $H$  is an extension of the rational group  $B$  by the rational group  $A$ :

$$(6.3) \quad 1 \rightarrow A \rightarrow H \rightarrow B \rightarrow 1.$$

We shall divide the argument into two cases according as  $B \cong \mathbb{Z}$  or not.

CASE 1.  $B \cong \mathbb{Z}$ . Let  $t$  be a generator of  $B$ , and let  $\phi \in \text{Auto}(A)$  be defined by  $\phi(a) = tat^{-1}$ . An automorphism  $\phi$  is described as  $\phi(a) = \xi a$  for some rational number  $\xi$ . (Here we use additive notation for the group  $A$ .) If  $\xi < 0$ , then there is no bi-invariant order on  $H$ , and we are done. If  $\xi > 0$ , then there is an injective homomorphism  $\iota : H \rightarrow \text{Aff}_+(\mathbb{R})$  to the group of the order preserving affine transformations of  $\mathbb{R}$  defined as follows. The exact sequence (6.3) is split, and we can write any element of  $H$  as a pair  $(a, t^n)$ , where  $a \in A$ , in a standard way. The injection  $\iota$  is defined by

$$\iota(a, t^n) = \begin{pmatrix} \xi^n & a \\ 0 & 1 \end{pmatrix},$$

The affine  $H$ -action is free at any irrational point of  $\mathbb{R}$ , and various choices of irrational base points give various left orders on  $H$ . But the group  $H$  has exactly

4 left orders, all induced from the left orders of  $G$ . (Otherwise, the cardinality of  $LO(G)$  would be  $> 2^{n+1}$ .) A contradiction.

CASE 2.  $B \not\cong \mathbb{Z}$ . First we consider the case where  $A \not\cong \mathbb{Z}$ . Assume, for contradiction, that one of the 4 orders, say  $\lambda_0$ , is a bi-order. Let  $S_0$  be a characteristic positive set of cardinality 2. Then, as is well known (e.g, [10]), the dynamical realization  $\rho_{\lambda_0}$  of the bi-order  $\lambda_0$  based at  $x_0$  is almost free, i.e, for any  $g \in H$ , either  $\rho_{\lambda_0}(g)x \geq x$  for any  $x \in \mathbb{R}$ , or  $\rho_{\lambda_0}(g)x \leq x$  for any  $x \in \mathbb{R}$ . That is, for any  $g \in H$ , the left order determined by the order of the orbits of  $\rho_{\lambda_0}(g)(x_0)$  has  $S_0$  in its positive cone.

Since  $S_0$  is a characteristic positive set, and since the orbit  $\rho_{\lambda_0}(G)(x_0)$  is dense by the assumption  $A \not\cong \mathbb{Z}$ , there is a centralizer  $h_g$  of  $\rho_{\lambda_0}(H)$  in  $\text{Homeo}_+(\mathbb{R})$  such that  $h_g(x_0) = gx_0$ . The group of centralizers acts freely and minimally on  $\mathbb{R}$ . This shows that the action  $\rho_\lambda$  is free, which implies that the group  $H$  is abelian. Therefore we have an isomorphism  $H \cong A \times B$ .

Let  $\iota_1 : A \rightarrow \mathbb{R}$  and  $\iota_2 : B \rightarrow \mathbb{R}$  be injective homomorphisms. Projecting the image of the diagonal embedding  $(\iota_1, \iota_2) : A \times B \rightarrow \mathbb{R}^2$  along various irrational directions yields various left orders on  $H$ . A contradiction.

Next consider the case where  $A \cong \mathbb{Z}$ . In this case, there is a homomorphism  $\sigma : H \rightarrow \{\pm 1\}$  such that  $hah^{-1} = \sigma(h)a$  ( $h \in H, a \in A$ ). If  $\sigma$  is nontrivial, then  $H$  does not admit a bi-order. If  $\sigma$  is trivial, then  $H \cong A \times B$ , and the same argument as before leads to a contradiction. The proof of (I) is complete.

Now we proceed to the proof of (II). We only consider the case where  $G$  is given by

$$(6.4) \quad 1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1,$$

since the other case can be reduced to this case. The group  $A$  and  $B$  is rational, and  $G$  has no bi-invariant order. We shall show that  $A$  is  $\lambda$ -convex for any  $\lambda \in LO(G)$ . This is sufficient since then there are only 4 left orders on  $G$ . Let  $\phi : G \rightarrow \text{Aut}(A)$  be a homomorphism defined by  $\phi(g) = gag^{-1}$  ( $g \in G, a \in A$ ). Since  $A$  is abelian, the homomorphism projects down to a homomorphism from  $B$ , which we denote by the same letter  $\phi$ . We have

$$\text{Aut}(A) = \{\xi \in \mathbb{Q} \mid \xi A = A\}.$$

If  $\phi(b) > 0$  for any  $b \in B \setminus \{e\}$ , then the lexicographic order defined by the sequence (6.4) is bi-invariant. To show this, consider its dynamical realization, and notice that the action of  $A$  on each gap of the minimal set is in the same direction. Therefore there is a  $\lambda$ -positive element  $b_0 \in B$  such that  $\phi(b_0^{-1}) = \xi$  is negative. Let  $a \in A$  be an arbitrary  $\lambda$ -positive element. Then we have  $\xi a <_\lambda e$ , and therefore  $ab_0 = b_0(\xi a) < b_0$ . This shows that  $a[e, b_0]_\lambda \subset [e, b_0]_\lambda$ , where we set

$$[h, h']_\lambda = \{g \in G \mid h <_\lambda g <_\lambda h'\}.$$

Especially  $a <_\lambda b_0$ . Likewise we can show  $-b_0 <_\lambda a$  for any  $\lambda$ -negative  $a$ . That is,  $A \subset [-b_0, b_0]_\lambda$ . Since a  $\lambda$ -positive element  $b_0$  for which  $\phi(b_0^{-1}) < 0$  can be chosen arbitrarily small, we have shown that  $A$  is  $\lambda$ -convex.  $\square$

REMARK 6.3. Let  $(I_i, \mathcal{M}_i)$  be the  $i$ -th internal pair associated with the maximal convex sequence (6.1) of a Tararin group  $G$ . The next subgroup  $G_{i+1}$  leaves the gap  $I_{i+1}$  of  $\mathcal{M}_i$  in  $I_i$  invariant. But because  $G_{i+1}$  is a normal subgroup of  $G_i$ , it leaves all the iterates of  $I_{i+1}$  invariant. By Lemma 3.4, these are the only gaps of

$\mathcal{M}_i$ . Therefore  $G_{i+1}$  acts trivially on  $\mathcal{M}_i$ . That is, there is an induced action of  $G_i/G_{i+1}$  on  $\mathcal{M}_i$ . If  $\mathcal{M}_i$  is discrete, then  $G_i/G_{i+1} \cong \mathbb{Z}$ , and the action on  $\mathcal{M}_i$  is by translation. Assume  $\mathcal{M}_i$  is locally Cantor. Let  $\mathcal{R}_i$  be the quotient space obtained by  $I_i$  by collapsing each gap of  $\mathcal{M}_i$  to a point. It is homeomorphic to  $\mathbb{R}$ . The quotient group  $G_i/G_{i+1}$  acts on  $\mathcal{R}_i$  minimally and freely. The whole action of  $G$  on  $\mathbb{R}$  is a “pileup” of translations. Any left order is discrete if and only if the last group  $G_n$  is isomorphic to  $\mathbb{Z}$ .

## 7. Maximal convex sequence

We shall raise one more example of isolated orders whose height is as big as possible. Let  $B_n$  be the braid group of  $n$  strings, with the standard generators  $\sigma_1, \dots, \sigma_{n-1}$ . Define

$$z_1 = \sigma_1 \cdots \sigma_{n-1}, \quad z_2 = \sigma_2 \cdots \sigma_{n-1}, \quad \dots, \quad z_{n-2} = \sigma_{n-2} \sigma_{n-1}, \quad z_{n-1} = \sigma_{n-1},$$

and  $y_i = z_i^{(-1)^{i-1}}$ . Let  $P_n$  be the subsemigroup of  $B_n$  generated by  $y_i$ 's. Based upon a result of P. Dehornoy [1], T. V. Dubrovina and N. I. Dubrovin [2] has shown a remarkable fact that  $P_n \sqcup P_n^{-1} = B_n \setminus \{e\}$ . The left order  $\lambda_n$  whose positive cone is  $P_n$  is called the *Dubrovina-Dubrovin order*. Since  $S = \{y_1, \dots, y_{n-1}\}$  generates  $P_n$ , the order  $\lambda_n$  is isolated with characteristic positive set  $S$ . Moreover  $\lambda_n$  can be defined lexicographically as a twist of the Dehornoy order [1], and the subgroups

$$B_{n-k}^* = \langle y_k, \dots, y_{n-1} \rangle = \langle \sigma_k, \dots, \sigma_{n-1} \rangle$$

are  $\lambda_n$ -convex. Since  $|S| = n - 1$ , they are the only convex subgroups, and the maximal convex sequence is given by

$$(7.1) \quad B_n > B_{n-1}^* > \cdots > B_2^* > \{e\}.$$

The height of  $\lambda_n$  is  $n - 2$ . The order  $\lambda_n$  is discrete since  $B_2^* \cong \mathbb{Z}$ . The  $i$ -th minimal set  $\mathcal{M}_i$  of the  $i$ -th internal pair  $(I_i, \mathcal{M}_i)$  is locally Cantor, since each term in (7.1) is not a normal subgroup of the previous term.

We shall construct a new isolated order of  $B_3$  in Section 10.

For an isolated order  $\lambda \in LO(G)$ , we can define the *maximal Tararin subgroup*  $G_i$  in its maximal convex sequence

$$(7.2) \quad G > G_1 > \cdots > G_n > \{e\}.$$

For  $\lambda_n$ , the maximal Tararin subgroup is  $B_2^* \cong \mathbb{Z}$ , and its height is 0. We shall raise questions about the isolated orders of non Tararin groups.

QUESTION 7.1. Is there a non Tararin group with an isolated order whose maximal Tararin subgroup has height  $\geq 1$ ?

QUESTION 7.2. Is there a non Tararin group with an isolated and indiscrete order?

There is a sufficient condition for a group to be Tararin in terms of an isolated order on it.

PROPOSITION 7.3. *If the maximal convex sequence of an isolated order  $\lambda \in LO(G)$  is subnormal,<sup>2</sup> then  $G$  is a Tararin group.*

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<sup>2</sup>each term is a normal subgroup of the previous term

PROOF. The proof is an induction on the height of  $\lambda$ . For height 0, this is true by Theorem 4.1. Assume the height is  $\geq 1$  and consider the maximal convex sequence of  $\lambda$ :

$$(7.3) \quad G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{e\}.$$

By the induction hypothesis,  $G_1$  is a Tararin group and the subsequence of (7.3) that begins with  $G_1$  is the unique rational series in Theorem 6.2. Since each  $G_i$   $2 \leq i \leq n$ , is a characteristic subgroup of  $G_1$  and since  $G_1$  is a normal subgroup of  $G$ ,  $G_i$  is a normal subgroup of  $G$ . By virtue of lemmas 5.1 and 5.2, the order induced from  $\lambda$  on  $G/G_1$  is isolated. Therefore  $G/G_1$  is a rational group, by virtue of Theorem 4.1. That is, the sequence (7.3) is a rational series. Moreover by the same argument as in the proof of Theorem 6.2. one can show that  $G/G_2$  admits no bi-invariant order.  $\square$

COROLLARY 7.4. *Let  $\lambda \in LO(G)$  be isolated of height 1. If the minimal set of the dynamical realization is discrete, then  $G$  is a Tararin group.*

PROOF. If a minimal set is discrete, then we get a surjective homomorphism  $\phi : G \rightarrow \mathbb{Z}$  and its kernel is a convex subgroup. By the previous proposition,  $G$  is a Tararin group.  $\square$

EXAMPLE 7.5. The above corollary does not hold if we remove the condition that  $\lambda$  is height 1. Let us construct an example of isolated order  $\lambda \in LO(G)$  of height 2 with discrete minimal set, where  $G$  is non Tararin. We start with the braid group  $B_3$ . The subsemigroup  $P$  generated by  $y_1 = \sigma_1\sigma_2$  and  $y_2 = \sigma_2^{-1}$  is the positive cone of the Dubrovina-Dubrovin order  $\lambda_3$ . The group  $B_3$  is described as

$$B_3 = \langle y_1, y_2 \mid y_2 y_1^2 y_2 = y_1 \rangle.$$

There is an automorphism  $\phi$  of  $B_3$  which satisfies  $\phi(y_1) = y_1^{-1}$  and  $\phi(y_2) = y_2^{-1}$ . Therefore if we define a group  $G$  by

$$G = \langle x, y_1, y_2 \mid y_2 y_1^2 y_2 = y_1, x y_1 x^{-1} = y_1^{-1}, x y_2 x^{-1} = y_2^{-1} \rangle,$$

then  $B_3$  is a subgroup of  $G$  [4]. Let  $\hat{P}$  be the subsemigroup of  $G$  generated by  $x$  and  $P$ . Then since  $xP = P^{-1}x$ , we have  $G \setminus \{e\} = \hat{P} \sqcup \hat{P}^{-1}$ . The left order  $\lambda$  on  $G$  determined by  $\hat{P}$  has  $B_3$  as a  $\lambda$ -convex normal subgroup. In fact,

$$B_3^{-1}x = (P \sqcup P^{-1} \sqcup \{e\})x = Px \sqcup P^{-1}x \sqcup \{x\} = Px \sqcup xP \sqcup \{x\} \subset \hat{P},$$

and likewise  $B_3^{-1}x^{-1} \in \hat{P}^{-1}$ , which means  $x^{-1} <_\lambda B_3 <_\lambda x$ . Since  $G/B_3 \cong \mathbb{Z}$ , the minimal set associated to  $\lambda$  is discrete. The dynamics of  $\lambda$  is as depicted in Figure 1.

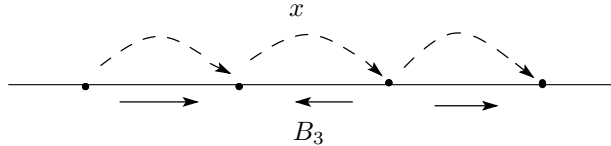


FIGURE 1. The dotted points form the minimal set  $\mathcal{M}$ . The element  $x$  moves these points one to the right. The intervals bounded by the points are invariant by  $B_3$ . The actions of  $B_3$  are opposite in neighbouring intervals, showing the stability of the action.

A. Navas [10] have defined the Conradian soul  $C_\lambda$  for any  $\lambda \in LO(G)$ . Let us recall it briefly. A left order  $\lambda \in LO(G)$  of a group  $G$  is called *Conradian* if we have  $g^{-1}hg^2 >_\lambda e$  whenever  $g, h \in G$  and  $h >_\lambda e$ . Thus a bi-invariant order is Conradian. Given an action of  $G$  on  $\mathbb{R}$ , a point  $x \in \mathbb{R}$ , is called *resilient* if there are an element  $h$  of the stabilizer of  $x$  and a point  $y \in Gx \setminus \{x\}$  such that  $h^n y \rightarrow x$  as  $n \rightarrow \infty$ . It is shown [10] that  $\lambda \in LO(G)$  is Conradian if and only if the dynamical realization of  $\lambda$  admits no resilient point.

For a general left order  $\lambda \in LO(G)$ , a subgroup  $H < G$  is called  $\lambda$ -Conradian if the restriction of  $\lambda$  to  $H$  is Conradian. The *Conradian soul*  $C_\lambda$  of  $\lambda$  is defined to be the maximal convex Conradian subgroup. In other words, it is the union of all the convex Conradian subgroups. The following proposition is a consequence of [10], Proposition 4.1, which states that if a group  $G$  is non Tararin, a Conradian order of  $G$  can never be isolated. But we will give a proof based upon Proposition 7.3.

**PROPOSITION 7.6.** *If  $\lambda$  is isolated, the maximal Tararin subgroup of  $\lambda$  coincides with the Conradian soul of  $\lambda$ .*

**PROOF.** In the maximal convex sequece (7.2) of  $\lambda$ , let  $G_i$  be the maximal Tararin subgroup. It follows from Remark 6.3, that the dynamical realization of  $\lambda|_{G_i}$  is a pileup of translations, and cannot have a resilient point. Thus  $G_i$  is  $\lambda$ -Conradian. So it suffices to show that  $G_{i-1}$  is not  $\lambda$ -Conradian, that is, the dynamical realization of  $\lambda|_{G_{i-1}}$  admits a resilient point. It is no loss of generality to assume that  $i = 1$ . That is, we assume that  $G$  is not a Tararin group, while its maximal convex subgroup  $G_1$  is. By Proposition 7.3,  $G_1$  is not a normal subgroup of  $G$ . Then the minimal set  $\mathcal{M}$  is not discrete, and the action of  $G_1$  on  $\mathcal{M}$  is nontrivial. Choose  $g \in G_1$  which acts nontrivially on  $\mathcal{M}$ . Since  $G_1$  leaves invariant the gap  $I_1$  of  $\mathcal{M}$  containing the base point  $x_0$ , we have  $\text{Fix}(g) \cap \mathcal{M} \neq \emptyset$ . Then there are distinct points  $x, y \in \mathcal{M}$  such that  $g(x) = x$  and either  $g^n(y) \rightarrow x$  or  $g^{-n}(y) \rightarrow x$  as  $n \rightarrow \infty$ . Since the action of  $G$  on  $\mathcal{M}$  is minimal, the point  $y$  is accumulated by the orbit of  $x$ . This shows that the point  $x$  is resilient.  $\square$

## 8. Centerless groups with isolated left orders

Most of groups with isolated left orders admit nontrivial center, and the left orders come from circular orders of the quotient group. But this is not always the case.

**PROPOSITION 8.1.** *There is a centerless non Tararin group which admits isolated left orders.*

**PROOF.** First of all we shall construct a centerless Tararin group  $H$  of height 2 with discrete left orders. We shall define a group  $H_1$  which is a split extension of  $\mathbb{Z}[1/3]$  by  $\mathbb{Z}$ :

$$(8.1) \quad 1 \rightarrow \mathbb{Z} \rightarrow H_1 \rightarrow \mathbb{Z}[1/3] \rightarrow 1.$$

Let  $\epsilon : \mathbb{Z}[1/3] \rightarrow \{\pm 1\}$  be the unique nontrivial homomorphism. We have  $\text{Ker}(\epsilon) = 2\mathbb{Z}[1/3]$ . Define  $H_1$  to be  $\mathbb{Z} \times \mathbb{Z}[1/3]$  with multiplication

$$(n, \xi) \cdot (m, \eta) = (n + \epsilon(\xi)m, \xi + \eta).$$

There is an automorphism  $\Phi : H_1 \rightarrow H_1$  such that  $\Phi(n, \xi) = (n, -3\xi)$ , since  $\epsilon(-3\xi) = \epsilon(\xi)$ . Define  $H$  to be  $H_1 \times A$  with the multiplication

$$(g, a) \cdot (h, b) = (g \cdot \Phi^a(h), a + b).$$

We have a split exact sequence

$$1 \rightarrow H_1 \rightarrow H \xrightarrow{\sim} A \rightarrow 1.$$

The group  $A$  is identified with the subgroup  $\{e\} \times A$ . There is a rational series

$$H \triangleright H_1 \triangleright \mathbb{Z} \triangleright \{e\},$$

and it is routine to show that  $H_1$  and  $H/\mathbb{Z}$  are not bi-orderable. (We have used  $-$  in the construction.) Therefore  $H$  is a Tarsin group, and any left order  $\lambda_0$  of  $H$  is discrete. The subgroup  $A$  is  $\lambda_0$ -cofinal for any left order  $\lambda_0$  of  $H$ . Also it is easy to show that  $H$  is centerless.

Let  $B$  be a group isomorphic to  $\mathbb{Z}$  and  $A$  a proper nontrivial subgroup of  $B$ . Consider the amalgamated product  $G = H *_A B$ . T. Ito [5] has constructed an isolated left order  $\lambda$  on  $G$  which coincides with  $\lambda_0$  on  $H$ . It is easy to show that  $G$  is centerless and that  $G$  admits a subgroup isomorphic to the nonabelian free group. Therefore  $G$ , not being solvable, cannot be a Tarsin group.  $\square$

We suspect that the order  $\lambda$  constructed in Proposition 8.1 is of height 1, and the unique convex subgroup is malnormal.

## 9. Relations with circular orders

In this section, we provide preliminary facts which are needed in the construction of the next section.

DEFINITION 9.1. For a countable group  $\overline{G}$ , a map  $c : \overline{G}^3 \rightarrow \{0, 1, -1\}$  is called a *left invariant circular order of  $\overline{G}$*  if it satisfies the following conditions

- (1)  $c(g_1, g_2, g_3) = 0$  if and only if  $g_i = g_j$  for some  $i \neq j$ .
- (2) For any  $g_1, g_2, g_3, g_4 \in \overline{G}$ , we have

$$c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0.$$

- (3) For any  $g_1, g_2, g_3, g_4 \in \overline{G}$ , we have

$$c(g_4 g_1, g_4 g_2, g_4 g_3) = c(g_1, g_2, g_3).$$

DEFINITION 9.2. Given a finite set  $F$  of  $\overline{G}$ , a *positioning of  $F$  in  $S^1$*  is an equivalence class of injections  $\iota : F \rightarrow S^1$ , where two injections  $\iota$  and  $\iota'$  is said to be equivalent if there is an orientation preserving homeomorphism  $h$  of  $S^1$  such that  $\iota' = h\iota$ .

Given a left invariant circular order  $c$  of  $\overline{G}$ , the positioning of the three points set  $\{g_1, g_2, g_3\}$  is determined by the rule that  $g_1, g_2, g_3$  is positioned anticlockwise if  $c(g_1, g_2, g_3) = 1$ , and clockwise if  $c(g_1, g_2, g_3) = -1$ . By condition (2) of Definition 9.1, this is well defined. But (2) says more. One can show the following proposition by an easy induction on the cardinality of  $F$ .

PROPOSITION 9.3. *Given a left invariant circular order of  $\overline{G}$ , the positioning of any finite set  $F$  in  $S^1$  is determined.*  $\square$

Denote by  $CO(\overline{G})$  the set of all the left invariant circular orders. It is equipped with a totally disconnected compact metrizable topology, just as  $LO(\overline{G})$ . An *isolated left invariant circular order* is defined using this topology. If  $c \in CO(\overline{G})$  is isolated, then there is a finite set  $\overline{S}$  of  $\overline{G}$ , called a *determining set*, such that any left invariant circular order which gives the same positioning of  $\overline{S}$  in  $S^1$  is  $c$ .

Given  $c \in CO(\overline{G})$ , we define a dynamical realization  $\rho_c : \overline{G} \rightarrow \text{Homeo}^+(S^1)$  based at  $y_0 \in S^1$  as follows. Fix an enumeration of  $\overline{G}$ :  $\overline{G} = \{g_i \mid i \in \mathbb{N}\}$  such that  $g_1 = e$ . Define an embedding  $\iota : \overline{G} \rightarrow S^1$  inductively as follows. First, set  $\iota(g_1) = y_0$  and  $\iota(g_2) = y_0 + 1/2$ . If  $\iota$  is defined on  $\{g_1, \dots, g_n\}$ , then there is a connected component of  $S^1 \setminus \{\iota(g_1), \dots, \iota(g_n)\}$  where the point  $g_{n+1}$  should be embedded, by virtue of Proposition 9.3. Define  $\iota(g_{n+1})$  to be the midpoint of that interval. Using the injection  $\iota$ , we can define the action of  $\overline{G}$  on  $S^1$  just as in the case of left orders. The action is called the *dynamical realization of  $c$  based at  $y_0$*  and denoted by  $\rho_c$ . We shall raise fundamental properties of  $\rho_c$ . The proof is completely parallel to the case of left orders.

LEMMA 9.4. *The dynamical realization  $\rho_c$  is tight at the base point  $y_0$ , i.e., it is free at  $y_0$  and if  $I$  is a connected component of  $S^1 \setminus \text{Cl}(\rho_c(\overline{G})y_0)$ , then  $\partial I \subset \rho_c(\overline{G})y_0$ .*

LEMMA 9.5. *Two dynamical realizations obtained via different enumerations of  $G$  are mutually conjugate by an orientation and base point preserving homeomorphism of  $S^1$ .*

Let  $\mathcal{M}$  be a minimal set of the dynamical realization  $\rho_c$  of an isolated circular order  $c$ . It is shown by K. Mann and C. Rivas [8] that (unlike left orders)  $\mathcal{M}$  is always a proper subset of  $S^1$ . Summarizing with other properties, we get:

LEMMA 9.6. *If  $\overline{G}$  is not finite cyclic, the minimal set  $\mathcal{M}$  of the dynamical realization  $\rho_c$  of any isolated cyclic order  $c \in CO(L)$  is unique. It is either a finite set or a Cantor set.*

LEMMA 9.7. *If  $\overline{G}$  is not finite cyclic and  $c$  is isolated, then the base point  $y_0$  of the dynamical realization is contained in a gap  $I$  of the minimal set  $\mathcal{M}$ , the stabilizer  $\overline{G}_I$  of  $I$  is nontrivial, and there is no gap of  $\mathcal{M}$  other than the orbit of  $I$ .*

DEFINITION 9.8. Let  $c$  be a cyclic order of  $\overline{G}$ , isolated or not, and  $H$  a subgroup of  $\overline{G}$ .  $H$  is said to be  *$c$ -convex* if there exists an open interval  $I \subset \mathbb{R}$  such that  $\rho_c(\overline{G})y_0 \cap I = \rho_c(H)y_0 \cap I$ , equivalently if whenever  $c(h, e, h') = c(h, g, h') = 1$  for some  $h, h' \in H$  and  $g \in G$ , we have  $g \in H$ . The left order  $\lambda$  of  $H$  defined by the order of  $\rho_c(H)y_0 \subset I$  is said to be *induced from  $c$* . Formally,  $e <_\lambda h$  if there is  $h' \in H$  such that  $c(h', e, h) = 1$ .

Parallel to Lemma 5.1, we have:

LEMMA 9.9. *Let  $H$  be a subgroup of  $\overline{G}$ . For any  $\lambda_0 \in LO(H)$  and a  $\overline{G}$ -invariant circular order  $c_0$  on  $\overline{G}/H$ , there is a unique circular order  $c \in CO(\overline{G})$  such that  $H$  is  $c$ -convex,  $\lambda_0$  is induced from  $c$ , and that  $c(g_1, g_2, g_3) = c_0(g_1H, g_2H, g_3H)$  if  $g_iH$ 's are distinct.*

Such a circular order  $c$  is said to be *determined lexicographically by  $\lambda_0$  and  $c_0$* .

We make the following definition which is necessary in order to relate a left order to a circular order of a quotient group.



DEFINITION 9.10. A group  $G$  is said to satisfy *Condition (Z)* if its abelianization  $G/[G, G]$  is finitely generated, the center  $Z$  of  $G$  is isomorphic to  $\mathbb{Z}$ , and for any  $\lambda \in LO(G)$ ,  $Z$  is  $\lambda$ -cofinal.<sup>3</sup>

REMARK 9.11. There are groups without nontrivial center which admit isolated orders. An example is given in Section 8.

In the next section, we shall show that the braid group  $B_3$  satisfies Condition (Z).

Assume  $G$  satisfies Condition (Z) and let  $\lambda$  be an arbitrary left order of  $G$ . Consider the dynamical realization  $\rho_\lambda$  based at  $x_0$ . Then  $\rho_\lambda(Z)$  acts freely on  $\mathbb{R}$ , since  $Z$  is  $\lambda$ -cofinal. Therefore we obtain an action of the quotient group  $\overline{G} = G/Z$  on the quotient space  $\mathbb{R}/\rho_\lambda(Z)$  which is homeomorphic to  $S^1$ . The action is free at the projected image  $y_0$  of  $x_0$ , and we obtain a circular order on  $\overline{G}$  by looking at the orbit of  $y_0$ . More precisely we make the following definition.

DEFINITION 9.12. Assume  $G$  satisfies Condition (Z). For any  $\lambda \in LO(G)$ , a left invariant circular order  $c = c(\lambda)$  of the quotient group  $\overline{G} = G/Z$ , called the *projection of  $\lambda$* , is defined as follows. Given three distinct cosets  $Z$ ,  $g_1Z$  and  $g_2Z$ , choose representatives  $g_i$  of the cosets ( $i = 1, 2$ ) so that  $e <_\lambda g_i <_\lambda t$ , where  $t$  is the  $\lambda$ -positive generator of  $Z$ . Define  $c(Z, g_1Z, g_2Z) = 1$  (resp.  $= -1$ ) if  $g_1 <_\lambda g_2$  (resp.  $g_2 <_\lambda g_1$ ), and extend it to  $\overline{G}^3$  by the left invariance (3) and degeneracy (1) of Definition 9.1.

THEOREM 9.13. *Let  $G$  satisfy Condition (Z) and let  $\lambda \in LO(G)$ . If  $c(\lambda)$  is isolated, then  $\lambda$  is isolated.*

PROOF. Assume  $c(\lambda)$  is isolated, and let  $\overline{S} = \{Z, s_1Z, \dots, s_rZ\}$  be a finite determining set of  $c(\lambda)$ . Choose representatives  $s_i$  of the cosets  $s_iZ$  so that they satisfy

$$(9.1) \quad e <_\lambda s_1 <_\lambda \dots <_\lambda s_r <_\lambda t,$$

after permuting the elements if necessary. Let  $s_0 = e$  and  $s_{r+1} = t$  and

$$(9.2) \quad S = \{s_i^{-1}s_{i+1} \mid i = 0, \dots, r\}.$$

If  $G$  is not perfect, add finite  $\lambda$ -positive elements whose cosets generate  $G/[G, G]$  to  $S$ . Precisely, we enlarge the finite determining set  $\overline{S}$  so that the set  $S$  determined by (9.1) and (9.2) contains those added elements.

Let us show that  $S$  is a characteristic positive set for  $\lambda$ . First notice that  $S$  is contained in the positive cone of  $\lambda$  by (9.1). Let  $\lambda' \in LO(G)$  be an arbitrary left order whose positive cone contains  $S$ . Let  $\rho_\lambda$  (resp.  $\rho_{\lambda'}$ ) be the dynamical realization of  $\lambda$  (resp.  $\lambda'$ ) based at  $x_0$ . Recall the generator  $t$  of  $Z$ . One may assume that  $\rho_\lambda(t) = \rho_{\lambda'}(t) = \tau$  is the translation by one, by changing  $\rho_\lambda$  and  $\rho_{\lambda'}$  in their conjugacy classes. Denote by  $p : \mathbb{R} \rightarrow \mathbb{R}/\langle \tau \rangle = S^1$  the projection map and let  $y_0 = p(x_0)$ . Now  $c(\lambda)$  and  $c(\lambda')$  determine the same positioning in  $S^1$  of the determining set  $\overline{S}$ . Hence by the assumption of the lemma we have  $c(\lambda) = c(\lambda')$ . Let  $\overline{\rho}_\lambda$  (resp.  $\overline{\rho}_{\lambda'}$ ) be the project down of  $\rho_\lambda$  (resp.  $\rho_{\lambda'}$ ) to  $S^1$ . Precisely, define  $\overline{\rho}_\lambda : \overline{G} \rightarrow \text{Homeo}_+(S^1)$  (resp.  $\overline{\rho}_{\lambda'}$ ) by requiring

$$\overline{\rho}_\lambda(gZ)p = p\rho_\lambda(g), \quad \forall g \in G$$

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<sup>3</sup>I.e., for any  $g \in G$ , there is  $t, t' \in Z$  such that  $t <_\lambda g <_\lambda t'$ .

(resp.  $\bar{\rho}_{\lambda'}(gZ)p = p\rho_{\lambda'}(g)$ ,  $\forall g \in G$ ).

Then  $\bar{\rho}_\lambda$  (resp.  $\bar{\rho}_{\lambda'}$ ) is tight at  $y_0$ , and is topologically conjugate to the dynamical realization of  $c(\lambda)$  (resp.  $c(\lambda')$ ). Thus there is an orientation preserving homeomorphism  $\bar{h}$  of  $S^1$  such that  $\bar{h}(y_0) = y_0$  and that

$$\bar{h}\bar{\rho}_{\lambda'}(gZ) = \bar{\rho}_\lambda(gZ)\bar{h}, \quad \forall g \in G.$$

Let  $h \in \text{Homeo}_+(\mathbb{R})$  be a lift of  $\bar{h}$  such that  $h(x_0) = x_0$ . We have the equality of subsets:

$$(9.3) \quad h\rho_{\lambda'}(gZ)(x) = \rho_\lambda(gZ)h(x), \quad \forall x \in \mathbb{R}.$$

Since  $\rho_\lambda$  and  $\rho_{\lambda'}$  acts freely on the point  $h(x_0) = x_0$ , there is a bijective map  $\phi : G \rightarrow G$  such that

$$h\rho_{\lambda'}(g)(x_0) = \rho_\lambda(\phi(g))h(x_0).$$

It satisfies  $\phi(g) \in gZ$  for any  $g \in G$ . Let us show that

$$(9.4) \quad h\rho_{\lambda'}(g)(x) = \rho_\lambda(\phi(g))h(x), \quad \forall x \in \mathbb{R}.$$

Consider the two subsets  $\Gamma_g$  and  $\Gamma'_g$  of  $\mathbb{R}^2$  defined by

$$\Gamma_g = \{(x, y) \mid y \in \rho_\gamma(gZ)(x)\}, \quad \Gamma'_g = \{(x, y) \mid y \in \rho_{\gamma'}(gZ)(x)\}.$$

The set  $\Gamma_g$  consists of curves  $C_n$ ,  $n \in \mathbb{Z}$ , that are the graphs of functions

$$x \rightarrow \rho_\gamma(gt^n)x = \tau^n(\rho_\gamma(g))(x).$$

Thus  $C_n$  is the image of  $C_0$  by the vertical translation by  $n$ . Likewise  $\Gamma'_g$  consists of curves  $C'_n$ . Now by (9.3), the map  $h \times h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  maps  $\Gamma'_g$  to  $\Gamma_g$ . It maps the point  $(x_0, \rho_{\lambda'}(g)x_0)$  of  $\Gamma'_g$  to the point  $(h(x_0), h\rho_{\lambda'}(g)x_0) = (h(x_0), \rho_\lambda(\phi(g))h(x_0))$ . Therefore it maps the connected component of  $(x_0, \rho_{\lambda'}(g)x_0)$  in  $\Gamma'_g$  to the connected component of  $(h(x_0), \rho_\lambda(\phi(g))h(x_0))$  in  $\Gamma_g$ . But this is exactly (9.4).

Now (9.4) can be rewritten as the identity of maps:

$$(9.5) \quad h\rho_{\lambda'}(g) = \rho_\lambda(\phi(g))h,$$

which shows that  $\phi$  is an automorphism of  $G$ . Moreover since  $\phi$  maps a coset of  $Z$  onto itself, it is of the form  $\phi(g) = gt^{\alpha(g)}$  for some homomorphism  $\alpha : G \rightarrow \mathbb{Z}$ . The proof will be complete if we show that  $\alpha$  is trivial, since then  $\phi$  is the identity, that is,  $\phi'_\lambda$  is topologically conjugate to  $\phi_\lambda$  by the homeomorphism  $h$ , which implies that  $\lambda' = \lambda$ .

This is true if  $G$  is perfect. Let us consider the general case. Since  $\bar{h}$  maps  $\bar{\rho}_{\lambda'}(\bar{S})(y_0)$  bijectively to  $\bar{\rho}_\lambda(\bar{S})(y_0)$  in a way to preserve the circular orders,  $h$  maps the set  $\{\rho_{\lambda'}(s_i)(x_0) \mid i = 0, \dots, r\}$  bijectively to the set  $\{\rho_\lambda(s_i)(x_0) \mid i = 0, \dots, r\}$  in an order preserving way, where the elements  $s_i$  are from (9.1). That is, we have

$$h(\rho_{\lambda'}(s_i)(x_0)) = \rho_\lambda(s_i)(x_0) = \rho_\lambda(s_i)(h(x_0)).$$

Comparing this with (9.5) and noticing that  $\rho_\lambda$  is free at  $h(x_0) = x_0$ , we get  $\phi(s_i) = s_i$ . Thus the automorphism  $\phi$  must be the identity on the characteristic positive set  $S$  determined by (9.2). Equivalently,  $\alpha$  vanishes on  $S$ . Since  $S$  contains elements whose cosets generate  $G/[G, G]$ ,  $\alpha$  must be trivial.  $\square$

**REMARK 9.14.** In the above proof, it is not completely necessary that the set  $S$  determined by (9.1) and (9.2) contains the elements whose cosets generate  $G/[G, G]$ . For example, if  $G/[G, G]$  is infinite cyclic, it suffices for  $S$  to contain at least one element from  $G \setminus [G, G]$ .

### 10. Isolated left orders on $B_3$

In this section, we construct two isolated left orders on the braid group  $B_3$ , using Theorem 9.13. One is the reciprocal  $\lambda_3^*$  of the Dubrovina-Dubrovin order  $\lambda_3$ , and the other is not an automorphic image<sup>4</sup> of  $\lambda_3$ . Both stems from a theorem of Y. Matsuda [9]. The group  $B_3$  has the following representations.

$$\begin{aligned} B_3 &= \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \\ &= \langle y_1, y_2 \mid y_2 y_1^2 y_2 = y_1 \rangle \\ &= \langle a, b \mid a^2 = b^3 \rangle, \end{aligned}$$

where the generators are related by

$$y_1 = \sigma_1 \sigma_2, \quad y_2 = \sigma_2^{-1}, \quad a = y_1^{-1} y_2, \quad b = y_1^{-1}.$$

The reciprocal  $\lambda_3^*$  of the Dubrovina-Dubrovin order  $\lambda_3$  is the unique left order on  $B_3$  which satisfies  $y_1 < e$  and  $y_2 < e$ , equivalently  $e < a < b$ . Henceforth in this section we denote

$$G = \langle a, b \mid a^2 = b^3 \rangle.$$

Its centralizer  $Z$  is the infinite cyclic group generated by  $t = a^2 = b^3$ .

**PROPOSITION 10.1.** *The group  $G$  above satisfies Condition (Z) of the previous section.*

**PROOF.** We have only to show that for any  $\lambda \in LO(G)$ ,  $Z$  is  $\lambda$ -cofinal. It is no loss of generality to assume  $a > e$  (which is equivalent to  $b > e$ ). Any element  $g$  of  $G$  can be represented as a word:

$$g = a^{i_1} b^{j_1} \dots a^{i_r} b^{j_r},$$

where  $i_1$  and  $j_r$  may be 0. But we can write  $a^{i_\nu} = t^{n_\nu} a^{-1}$  and  $b^{j_\nu} = t^{m_\nu} b^{-1}$  or  $b^{j_\nu} = t^{m_\nu} b^{-2}$ . If we move iterates of  $t$  to the beginning of the word, we get an expression

$$g = t^N a^{i'_1} b^{j'_1} \dots a^{i'_r} b^{j'_r},$$

where all the numbers  $i'_\nu$  and  $j'_\nu$  are nonpositive. Since  $a^{i'_1} b^{j'_1} \dots a^{i'_r} b^{j'_r} \leq e$ , we get that  $g \leq t^N$ . Likewise if we write  $a^{i_\nu} = t^{n'_\nu} a$  and  $b^{j_\nu} = t^{m'_\nu} b$  or  $b^{j_\nu} = t^{m'_\nu} b^2$ , we can show that there is  $N'$  such that  $t^{N'} \leq g$ .  $\square$

The quotient group  $\overline{G} = G/Z$  is represented by

$$\overline{G} = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = e \rangle.$$

Let  $q : G \rightarrow \overline{G}$  be the natural projection defined by  $q(a) = \alpha$  and  $q(b) = \beta$ . There is an isomorphism  $\iota : \overline{G} \rightarrow PSL(2, \mathbb{Z})$  defined by

$$\iota(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \iota(\beta) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let us define a homomorphism  $\overline{\rho}_M : \overline{G} \rightarrow \text{Homeo}_+(S^1)$ , called the *modular representation*, as the composite

$$\overline{G} \xrightarrow{\iota} PSL(2, \mathbb{Z}) \subset PSL(2, \mathbb{R}) \subset \text{Homeo}_+(S^1).$$

---

<sup>4</sup>Given  $\phi \in \text{Aut}(G)$  and  $\lambda \in LO(G)$ , the element  $\phi^* \lambda \in LO(G)$  defined by  $g <_{\phi^* \lambda} g'$  if  $\phi(g) <_\lambda \phi(g')$  is called an automorphic image of  $\lambda$ . For  $B_3$ , the reciprocal  $\lambda_3^*$  is an automorphic image of  $\lambda_3$ .

It satisfies that

$$\begin{aligned} \bar{\rho}_M(\alpha)(0) &= \infty, \quad \bar{\rho}_M(\alpha)(\infty) = 0, \\ \bar{\rho}_M(\beta)(0) &= \infty, \quad \bar{\rho}_M(\beta)(\infty) = -1, \quad \bar{\rho}_M(\beta)(-1) = 0, \\ \text{and } \bar{\rho}_M(\alpha\beta)(0) &= 0. \end{aligned}$$

See Figure 2. We have

$$(\text{rot}(\bar{\rho}_M(\alpha)), \text{rot}(\bar{\rho}_M(\beta)), \text{rot}(\bar{\rho}_M(\alpha\beta))) = (1/2, 1/3, 0),$$

where  $\text{rot}(\cdot)$  is the rotation number.

Given  $k > 1$ , a representation  $\rho' : \bar{G} \rightarrow \text{Homeo}_+(S^1)$  is called a  $k$ -fold lift of  $\bar{\rho}_M$  if  $p_k \rho'(g) = \bar{\rho}_M(g)$  holds for any  $g \in \bar{G}$ , where  $p_k : S^1 \rightarrow S^1$  is the  $k$ -fold covering map. There is a  $k$ -fold lift of  $\bar{\rho}_M$  if and only if  $k \equiv \pm 1 \pmod{6}$ , and such a lift is unique if it exists. We shall denote it by  $\bar{\rho}_M^{(k)}$ . As for the rotation number of  $\bar{\rho}_M^{(k)}$  for  $k \equiv -1 \pmod{6}$ , we have

$$(\text{rot}(\bar{\rho}_M^{(k)}(\alpha)), \text{rot}(\bar{\rho}_M^{(k)}(\beta)), \text{rot}(\bar{\rho}_M^{(k)}(\alpha\beta))) = (1/2, 2/3, 1/k).$$

A map  $\bar{h} : S^1 \rightarrow S^1$  is called a *monotone continuous degree one map* if it is continuous, degree one, and its lift  $h$  to  $\mathbb{R}$  satisfies  $h(x) \leq h(y)$  whenever  $x < y$ .

**THEOREM 10.2.** (*Y. Matsuda [9]*) (1) *If an action  $\rho : \bar{G} \rightarrow \text{Homeo}_+(S^1)$  satisfies*

$$(\text{rot}(\rho(\alpha)), \text{rot}(\rho(\beta)), \text{rot}(\rho(\alpha\beta))) = (1/2, 1/3, 0),$$

*then there is a monotone continuous degree one map  $\bar{h}$  such that  $\bar{\rho}_M(g)\bar{h} = \bar{h}\rho(g)$  for any  $g \in \bar{G}$ .*

(2) *If an action  $\rho : \bar{G} \rightarrow \text{Homeo}_+(S^1)$  satisfies*

$$(\text{rot}(\rho(\alpha)), \text{rot}(\rho(\beta)), \text{rot}(\rho(\alpha\beta))) = (1/2, 2/3, 1/5),$$

*then there is a monotone continuous degree one map  $\bar{h}$  such that  $\rho_M^{(5)}(g)\bar{h} = \bar{h}\rho(g)$  for any  $g \in \bar{G}$ .*

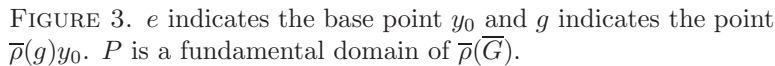
**REMARK 10.3.** The original statement in [9] is the existence of semi-conjugacies. But we can restate it using continuous monotone degree one map since the actions  $\bar{\rho}_M$  and  $\bar{\rho}_M^{(5)}$  are minimal.

**REMARK 10.4.** In [9], it is shown that among all  $k > 1$  with  $k \equiv \pm 1 \pmod{6}$ ,  $k = 5$  is the unique one to have the rotation number rigidity as above.

As the first application of Theorem 10.2, we shall give a dynamical proof of the Dubrovina-Dubrovina theorem for  $G = B_3$ . The proof serves as a prototype of the slightly more intriguing argument for Theorem 10.8.

**THEOREM 10.5.** *There is a unique left order  $\lambda \in LO(G)$  which satisfies  $e <_\lambda a <_\lambda b$ .*

**PROOF.** First we shall show the existence of  $\lambda$  in the theorem. We begin with the study of the dynamics of the modular representation  $\bar{\rho}_M$ . See Figure 2. The element  $\bar{\rho}_M(\alpha)$  is the  $1/2$ -rotation around  $i$ , and  $\bar{\rho}_M(\beta)$  the  $1/3$ -rotation around  $\omega = (-1 + \sqrt{-3})/2$ . The element  $\bar{\rho}_M(\alpha\beta)$  is a parabolic transformation which fixes the point 0 and moves points on  $S^1 \setminus \{0\}$  clockwise, as is depicted in Figure 2. The Fuchsian group  $\bar{\rho}_M(\bar{G})$  is of the first kind, that is, the action  $\bar{\rho}_M$  is minimal. We set the base point  $y_0$  to be equal to 0.



Let us define another Fuchsian representation  $\bar{\rho} : \overline{G} \rightarrow \text{Homeo}_+(S^1)$ , a deformation of  $\bar{\rho}_M$ . Choose a point  $\omega'$  on the geodesic which passes through  $i$  and  $\omega$ , but slightly farther than  $\omega$  from  $i$ :  $d(\omega', i) > d(\omega, i)$ . See Figure 3. We set  $\bar{\rho}(\alpha)$  to be the same as  $\bar{\rho}_M(\alpha)$ , the  $1/2$ -rotation around  $i$ , and  $\bar{\rho}(\beta)$  the  $1/3$ -rotation around  $\omega'$ . Again we put the base point  $y_0 = 0$ . The Fuchsian group  $\bar{\rho}(\overline{G})$  is of the second kind. Let  $\Lambda$  be its limit set.  $\Lambda$  is the (unique) minimal set of the action  $\bar{\rho}$  and is homeomorphic to the Cantor set. The fundamental domain  $P$  in Figure 3 (which is to be a closed set in the closed disk) and its iterates are disjoint from  $\Lambda$ . In particular  $y_0$  (depicted as  $e$  in Figure 3) is contained in a gap of  $\Lambda$ . The element  $\bar{\rho}(\alpha\beta)$  is hyperbolic whose axis is depicted by the dotted line in Figure 3. Let  $\sigma_-$  ( $\sigma_+$ )

be the source (resp. sink) of  $\bar{\rho}(\alpha\beta)$ . The open interval <sup>5</sup>  $(\sigma_-, \sigma_+)$  which contains  $y_0$  is a gap of  $\Lambda$ . Therefore  $\bar{\rho}$  acts freely at  $y_0$ . Notice that  $\bar{\rho}(\alpha\beta)$  moves points in the gap  $(\sigma_-, \sigma_+)$  anti-clockwise. The representation  $\bar{\rho}$  satisfies the condition of Theorem 10.2 (1), and the monotone continuous degree one map  $\bar{h}$  established in (1) maps each gap of  $\Lambda$  to a point. Since  $\bar{h}$  maps the fixed points of  $\bar{\rho}(\alpha\beta)$  to a fixed point of  $\bar{\rho}_M(\alpha\beta)$ , we have  $\bar{h}([\sigma_-, \sigma_+]) = \{y_0\}$ .

Denote by  $p : \mathbb{R} \rightarrow S^1$  the canonical projection and by  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  the translation by 1, a generator of the covering transformation. Define

$$\text{Homeo}_{\mathbb{Z}}\mathbb{R} = \{f \in \text{Homeo}_+(\mathbb{R}) \mid f\tau = \tau f\},$$

and denote by  $\pi : \text{Homeo}_{\mathbb{Z}}\mathbb{R} \rightarrow \text{Homeo}_+(S^1)$  the canonical projection. Recall the group

$$G = \langle a, b \mid a^2 = b^3 \rangle,$$

and the canonical projection  $q : G \rightarrow \bar{G} = G/Z$  defined by  $q(a) = \alpha$  and  $q(b) = \beta$ . Let us define a representation  $\rho : G \rightarrow \text{Homeo}_{\mathbb{Z}}\mathbb{R}$  as follows. The element  $\rho(a)$  is a lift of  $\bar{\rho}(\alpha)$  by  $\pi$  which satisfies  $\rho(a)^2 = \tau$ , and  $\rho(b)$  a lift of  $\bar{\rho}(\beta)$  which satisfies  $\rho(b)^3 = \tau$ . Thus  $\rho(t) = \tau$ , where  $t = a^2 = b^3$  is a generator of  $Z$ . There is a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{Homeo}_{\mathbb{Z}}\mathbb{R} \\ \downarrow q & & \downarrow \pi \\ \bar{G} & \xrightarrow{\bar{\rho}} & \text{Homeo}_+(S^1). \end{array}$$

Let  $x_0$  be any lift of  $y_0$ :  $p(x_0) = y_0$ . Then since  $\bar{\rho}$  acts freely at  $y_0$ ,  $\rho$  acts freely at  $x_0$ . Finally let  $\lambda$  be the left order of  $G$  which is determined by the order of the orbit of  $x_0$ . Then since  $x_0 < \rho(a)x_0 < \rho(b)x_0$  (which can readily be seen from Figure 3), we have  $e <_\lambda a <_\lambda b$ , finishing the proof of the existence.

Next let us show that the projection  $c(\lambda) \in CO(\bar{G})$  of  $\lambda \in LO(G)$  (Definition 9.12) is isolated. As can be seen from Figure 3, the cyclic order  $c(\lambda)$  satisfies

$$(10.1) \quad e < \alpha\beta < \alpha\beta\alpha < \alpha < \beta < \beta^2 < e.$$

The Fuchsian action  $\bar{\rho}$  is tight at  $y_0$ , and hence it is topologically conjugate to the dynamical realization of  $c(\lambda)$ . Consider an arbitrary cyclic order  $c' \in CO(\bar{G})$  which satisfies the same relation (10.1). Let  $\bar{\rho}'$  be its dynamical realization based at  $y_0$ . Then we have

$$(10.2) \quad y_0 < \bar{\rho}'(\alpha\beta)y_0 < \bar{\rho}'(\alpha\beta)\bar{\rho}'(\alpha)y_0 < \bar{\rho}'(\alpha)y_0 < \bar{\rho}'(\beta)y_0 < \bar{\rho}'(\beta^2)y_0 < y_0.$$

Notice that  $\bar{\rho}'(\alpha\beta)$  maps the interval  $[y_0, \bar{\rho}'(\alpha)y_0]$  into a proper subinterval of itself. Let

$$\sigma'_\pm = \lim_{n \rightarrow \pm\infty} \bar{\rho}'((\alpha\beta)^n)y_0.$$

Then the element  $\bar{\rho}'(\alpha\beta)$  leaves the interval  $[\sigma'_-, \sigma'_+]$  invariant and moves the interior points anticlockwise. Especially  $\text{rot}(\bar{\rho}'(\alpha\beta)) = 0$ . On the other hand, the last two relations in (10.2) implies that  $\text{rot}(\bar{\rho}'(\beta)) = 1/3$ , while  $\text{rot}(\bar{\rho}'(\alpha)) = 1/2$  is obvious. Therefore it satisfies the assumption of Theorem 10.2 (1), and there is a continuous monotone degree one map  $\bar{h}$  such that  $\bar{\rho}_M(g)\bar{h} = \bar{h}\bar{\rho}'(g)$  for any  $g \in \bar{G}$ . The map  $\bar{h}$  maps the fixed points  $\sigma'_\pm$  of  $\bar{\rho}'(\alpha\beta)$  to the unique fixed point  $y_0$  of  $\bar{\rho}_M(\alpha\beta)$ . Furthermore it maps the interval  $[\sigma'_-, \sigma'_+]$  to  $y_0$ , since  $\bar{\rho}_M(\alpha\beta)$  moves points in

<sup>5</sup>Given two points  $x, y \in S^1$ , we define  $(x, y) = \{t \in S^1 \mid x < t < y\}$ , where  $<$  is the anticlockwise circular order of  $S^1$ .

$S^1 \setminus \{y_0\}$  clockwise. In particule  $\bar{h}(y_0) = y_0$  since  $y_0 \in [\sigma'_-, \sigma'_+]$ . On the other hand, assume  $\bar{h}$  maps a point  $\bar{\rho}'(g)y_0$  to  $y_0$  for some  $g \in \bar{G}$ . Then we have

$$\bar{\rho}_M(g)y_0 = \bar{\rho}_M(g)\bar{h}(y_0) = \bar{h}\bar{\rho}'(g)y_0 = y_0.$$

Thus  $g$  must be contained in the stabilizer at  $y_0$  of the modular representation  $\bar{\rho}_M$ , that is, in the subgroup  $H = \langle \alpha\beta \rangle$ . By the tightness of the dynamical realization  $\bar{\rho}'$  at  $y_0$ , this shows that  $\bar{h}^{-1}(\{y_0\}) = [\sigma'_-, \sigma'_+]$ . In the language of circular orders, this amounts to the statement that  $H$  is a  $c'$ -convex subgroup, and the  $G$ -invariant circular order on  $G/H$  is determined by the orbit  $\bar{\rho}_M(\bar{G})y_0$ . On the other hand, the left order on the convex subgroup  $H$  induced from  $c'$  satisfies  $e < \alpha\beta$ . Therefore by Lemma 9.9, the circular order  $c'$  which satisfies (10.1) is unique. This shows  $c' = c(\lambda)$ .

Finally let us show that the left order  $\gamma$  constructed in the first part of the proof is uniquely determined by the relation  $e < a < b$ . This relation implies

$$(10.3) \quad e < abt^{-1} < abata^{-1} < a < b < b^2 < t,$$

where  $t = a^2 = b^3$ . In fact, the first inequality is obtained by

$$abt^{-1} = t^{-1}ab = a^{-1}b > e,$$

the second by

$$e < a \Rightarrow t^{-1} < t^{-1}a = at^{-1} \Rightarrow abt^{-1} < abata^{-1},$$

and the third by

$$b^2(ba^{-1}) = a < b < b^2 \Rightarrow ba^{-1} < e \Rightarrow abata^{-1} = aba^{-1} < a.$$

Now the proof of Theorem 9.13 shows that the relation (10.3) determines  $\gamma$ . Notice that for our  $G$ , the abelianization  $G/[G, G]$  is infinite cyclic, and the proof of Theorem 9.13 works if at least one element from  $G \setminus [G, G]$  is contained in the characteristic positive set.  $\square$

Now let us proceed to an application of Theorem 10.2 (2). First of all, there is a problem in considering the representation satisfying the condition (2), e.g.  $\bar{\rho}_M^{(5)}$ , because it does not lift to a representation from  $G$  to  $\text{Homeo}_{\mathbb{Z}}\mathbb{R}$  when we use the natural generator. So, instead of  $\bar{\rho}_M^{(5)}$ , we consider  $\bar{\rho}_M^{(5)} \circ \phi$ , where  $\phi \in \text{Aut}(\bar{G})$  is defined by  $\phi(\alpha) = \alpha$  and  $\phi(\beta) = \alpha\beta^2\alpha$ . Then since  $\phi(\alpha\beta) = (\alpha\beta)^{-1}$ , we have

$$(\text{rot}(\bar{\rho}_M^{(5)} \circ \phi)(\alpha), \text{rot}(\bar{\rho}_M^{(5)} \circ \phi)(\beta), \text{rot}(\bar{\rho}_M^{(5)} \circ \phi)(\alpha\beta)) = (1/2, 1/3, 4/5),$$

and Theorem 10.2 (2) transforms into the following form:

**THEOREM 10.6.** *If an action  $\bar{\rho} : \bar{G} \rightarrow \text{Homeo}_+(S^1)$  satisfies*

$$(\text{rot}(\bar{\rho}(\alpha)), \text{rot}(\bar{\rho}(\beta)), \text{rot}(\bar{\rho}(\alpha\beta))) = (1/2, 1/3, 4/5),$$

*then there is a monotone continuous degree one map  $\bar{h}$  such that  $(\bar{\rho}_M^{(5)} \circ \phi)(g)\bar{h} = \bar{h}\bar{\rho}(g)$  for any  $g \in \bar{G}$ .*

We suspect if  $f, g \in \text{Homeo}_+(S^1)$  satisfy  $f^2 = \text{id}$ ,  $\text{rot}(f) = 1/2$ ,  $g^3 = \text{id}$  and  $\text{rot}(g) = 1/3$ , then  $\text{rot}(fg)$  falls into the interval  $[4/5, 1]$ , i.e. the rotation number rigidity occurs at the extremal values of  $\text{rot}(fg)$  and not at any other values. But we do not pursue this problem in the present paper.

Theorem 10.6 suggests that the lift of the representation  $\bar{\rho}_M^{(5)} \circ \phi$  to  $G$  gives birth to an isolated order. But we begin by specifying the index 5 subgroup of  $G$



associated with  $\bar{\rho}_M^{(5)} \circ \phi$  and the left order induced from  $\lambda_3^*$ . Define a subgroup  $G_0$  of  $G$  by

$$G_0 = \langle A, B \rangle, \text{ where } A = at^2, B = ab^2a^{-1}t.$$

Clearly  $A$  and  $B$  satisfy

$$A^2 = B^3 = T, \text{ where } T = t^5.$$

LEMMA 10.7. *The homomorphism  $\Phi : G_0 \rightarrow G$  defined by  $\Phi(A) = a$  and  $\Phi(B) = b$  is an isomorphism. The subgroup  $G_0$  is normal and index 5 in  $G$ .*

PROOF.

The only nontrivial part of the first assertion is the injectivity of  $\Phi$ . There is an automorphism  $\psi$  of  $G_0$  which satisfies  $\psi(A) = A$  and  $\psi(A^{-1}BA) = B$ . So it suffices to show the injectivity of the map  $\Phi \circ \psi$  which maps  $at^2$  to  $a$  and  $b^2t$  to  $b$ . But this is a consequence of the uniqueness of the expression of  $g \in G$  as  $g = t^n a^{i_1} b^{j_1} \cdots a^{j_r} b^{j_r}$ , where  $n \in \mathbb{Z}$ ,  $i_1 = 0, 1$ ,  $i_\nu = 1$  ( $2 \leq \nu \leq r$ ),  $j_\nu = 1, 2$  ( $1 \leq \nu \leq r-1$ ) and  $j_r = 0, 1, 2$ . The details are left to the reader.

We shall omit the proof of the second assertion, since we do not use it explicitly.  $\square$

Recall that  $\lambda_3^*$  satisfies  $e < a < b$ . The restriction of  $\lambda_3^*$  to  $G_0$  satisfies

$$(10.4) \quad (AB)^5 T^{-4} < e < A < ((AB)^5 T^{-4})A.$$

In fact, we have  $(AB)^5 T^{-4} = (b^2 a^{-1})^5$ , and the first inequality follows from

$$b^3 a^{-1} = a < b \Rightarrow b^2 a^{-1} < e \Rightarrow (b^2 a^{-1})^5 < e,$$

and the last from

$$b^2 a^{-1} a > a \Rightarrow (b^2 a^{-1})^5 a > a \Rightarrow (b^2 a^{-1})^5 at^2 > at^2.$$

Our final result is the following.

THEOREM 10.8. *There is a unique left order  $\lambda_M \in LO(G)$  which satisfies*

$$(10.5) \quad (ab)^5 t^{-4} < e < a < (ab)^5 t^{-4} a.$$

The order  $\lambda_M$  is called the *Matsuda order*.

PROOF. The push forward of  $\lambda_3^*|_{G_0}$  by  $\Phi$  satisfies (10.5). Thus we only have to show the uniqueness. Let  $\rho$  be the dynamical realization of an arbitrary left order  $\lambda \in LO(G)$  which satisfies (10.5) based at  $x_0$ . Then we have

$$(10.6) \quad \rho((ab)^5 t^{-4})x_0 < x_0 < \rho(a)x_0 < (\rho((ab)^5 t^{-4}))(\rho(a)x_0).$$

The map  $\rho((ab)^{-5} t^4)$  maps the interval  $[x_0, \rho(a)x_0]$  into a proper subinterval of itself. Let  $\sigma_\pm = \lim_{n \rightarrow \pm\infty} \rho((ab)^{-5} t^4)^n(x_0)$ . The map  $\rho((ab)^{-5} t^4)$  leaves the interval  $[\sigma_-, \sigma_+]$  invariant and moves its interior points to the right. Let  $\bar{\rho}$  be the dynamical realization of the projection  $c(\lambda) \in CO(\bar{G})$  which satisfies  $p\rho(g) = \bar{\rho}(q(g))p$ , where  $p : \mathbb{R} \rightarrow S^1$  and  $q : G \rightarrow \bar{G}$  denotes the projection. It is based at  $y_0 = p(x_0)$ . Then we have  $\text{rot}(\bar{\rho}(\alpha\beta)) = 4/5$ , where  $\alpha = q(a)$  and  $\beta = q(b)$ . On the other hand, it is clear that  $\text{rot}(\bar{\rho}(\alpha)) = 1/2$  and  $\text{rot}(\bar{\rho}(\beta)) = 1/3$  by the relation  $a^2 = b^3 = t$ . Therefore by Theorem 10.6, there is a monotone continuous degree one map  $\bar{h}$  such that  $(\bar{\rho}_M^{(5)} \circ \phi)(g)\bar{h} = \bar{h}\bar{\rho}(g)$ .

Now  $(\bar{\rho}_M^{(5)} \circ \phi)((\alpha\beta)^{-5}) = \bar{\rho}_M^{(5)}((\alpha\beta)^5)$  is the lift of the parabolic transformation  $\bar{\rho}_M((\alpha\beta)^5)$ , and therefore it fixes five points, and moves the gap of the five points clockwise. The rest of the argument is parallel to the proof of Theorem 10.5:

we only outline the points. The map  $\bar{h}$  maps the interval  $[p(\sigma_-), p(\sigma_+)]$  to one of the five points, say  $y_0$ . If  $\bar{h}\bar{\rho}(g)(y_0) = y_0$  for some  $g \in \bar{G}$ , then  $g \in H = \langle (\alpha\beta)^5 \rangle$ . This shows that  $\bar{h}^{-1}(\{y_0\}) = [p(\sigma_-), p(\sigma_+)]$ . Thus  $H$  is  $c(\lambda)$ -convex, and the  $G$ -invariant circular order on  $G/H$  is determined by the orbit  $(\bar{\rho}_M\phi)(\bar{G})y_0$ . Also a linear order on  $H$  induced by  $c(\lambda)$  satisfies  $(\alpha\beta)^5 < e$ . Therefore, by the uniqueness of the lexicographically determined circular order (Lemma 9.9), the circular order  $c(\rho)$  on  $\bar{G}$  is isolated. This shows that the linear order  $\lambda$  on  $G$  is isolated. Moreover it is determined by (10.5).  $\square$

We suspect that there are no more isolated orders on  $B_3$  other than the automorphic images of the Dubrovina-Dubrovin order or the Matsuda order.

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